Classical spinning branes in curved backgrounds

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Abstract: The dynamics of a classical branelike object in a curved background is derived from the covariant stress-energy conservation of the brane matter. The world sheet equations and boundary conditions are obtained in the pole-dipole approximation, where nontrivial brane thickness gives rise to its intrinsic angular momentum. It is shown that intrinsic angular momentum couples to both, the background curvature and the brane orbital degrees of freedom. The whole procedure is manifestly covariant with respect to spacetime diffeomorphisms and world sheet reparametrizations. In addition, two extra gauge symmetries are discovered and utilized. The examples of the point particle and the string in 4 spacetime dimensions are analyzed in more detail. A particular attention is paid to the Nambu-Goto string with massive spinning particles attached to its ends.

Keywords: p-branes, Classical Theories of Gravity.

Contents

1. Introduction

The interest in studying extended objects in high energy physics began with the observation that meson resonances could be viewed as rotating relativistic strings. This model provided a successful explanation of Regge trajectories and Veneziano amplitudes. In later development of the idea, relativistic strings have been promoted to elementary building blocks of the known matter, and as such extensively studied.

A parallel line of research treated strings as linelike kink solutions in a field theory. Such is, for example, the Nielsen-Olesen vortex line solution of a Higgs type scalar electrodynamics [1]. The idea behind this approach is to try and describe bound states of quarks as flux tube solutions of the Standard model.

Whatever idea guides one to explore strings, or more generally branes, the general form of their classical dynamics may be needed. In this paper, we shall be concerned with the classical branelike kink configurations in an arbitrary Riemannian spacetime. We shall not specify the type of matter the branes are made of, but merely assume that such kink configurations exist. For simplicity, the dynamics of spacetime geometry is assumed to

be that of general relativity. In this setting, the stress-energy tensor of matter fields is symmetric, $T^{\mu\nu} = T^{\nu\mu}$, and covariantly conserved,

$$\nabla_\nu T^{\mu\nu} = 0 \,. \tag{1.1}$$

The covariant conservation law of the stress-energy tensor $T^{\mu\nu}$ is the starting point in our analysis of brane dynamics in curved spacetimes. The method we use is a generalization of the Mathisson-Papapetrou method for pointlike matter [2, 3]. It has already been exploited in ref. [4] for the study of stringlike objects in the lowest (single-pole) approximation. There, the world sheet effective equations of motion are obtained from the conservation equations (1.1) in the limit of an infinitely thin string. In this paper, we extend the analysis to the next level of approximation — the pole-dipole approximation. In this approximation, a nonzero thickness of the brane is taken into consideration. As we shall see, the additional degrees of freedom that thus appear account for the internal angular momentum of the brane.

The motivation for studying classical branelike matter in curved backgrounds is threefold. First, we believe it is useful to have fully covariant description of a classical p-brane with intrinsic angular momentum. We restrict ourselves to Riemannian spacetimes, but the analysis can be extended to include torsion. The dimensions of the brane (p) and the spacetime (D) remain arbitrary. Second, we find it interesting to try and extend the known Nambu-Goto string (described by the tension alone) to allow a nontrivial intrinsic angular momentum. A simple model of the kind could, at least, give us a clue to what kind of dynamics one could expect from the spinning string. Finally, our basic motivation for this work is proper preparation for treating strings in spacetimes with torsion. In the existing literature, the influence of torsion has been studied in the case of pointlike matter only. It has been suggested that the consistent treatment of the problem demands pole-dipole approximation [5, 6]. Naturally, we expect the same in the case of strings and higher branes.

The new results obtained in this paper can be summarized as follows. First, the Mathisson-Papapetrou method has been generalized for the treatment of higher branes in curved backgrounds. We have refined the method by developing a manifestly covariant decomposition of the stress-energy tensor into a series of δ-function derivatives. It has been shown that a truncation of the series is covariant with respect to both, spacetime diffeomorphisms and world sheet reparametrizations. In addition, two extra gauge symmetries are discovered and analyzed. The extra gauge fixing has been shown to define centre of mass of pointlike matter, and its generalization to central surface of mass in the case of branelike matter. Second, the fully covariant world sheet equations and boundary conditions of a p-brane in a D-dimensional Riemannian spacetime have been obtained in the pole-dipole approximation. The general brane dynamics turns out to depend on the effective $(p + 1)$-dimensional stress-energy tensor of the brane, and $(p + 1)$-dimensional currents corresponding to its internal angular momentum. We have utilized the discovered extra gauge freedom to show that charges corresponding to internal boosts can be gauged away. Finally, particles and strings in 4-dimensional spacetime have been analyzed in more

detail. It has been shown that the Nambu-Goto string can be generalized to include spinning matter on its ends, thereby providing a better model for meson resonances. In the case of pointlike matter, the known Papapetrou results are reproduced [3].

The layout of the paper is as follows. In section 2, we define covariant decomposition of the stress-energy tensor as a series of δ-function derivatives. We demonstrate the invariance with respect to spacetime diffeomorphisms, world sheet reparametrizations and two additional gauge transformations. The truncation of the series is shown to respect general covariance. In section 3, the world sheet equations and boundary conditions are derived in pole-dipole approximation. This is done by neglecting all but the first two terms in the decomposition of the stress-energy tensor, and using it in the conservation equations (1.1). In section 4, we analyze the symmetry properties of the free coefficients of our world sheet equations and boundary conditions. We identify the effective $(p + 1)$-dimensional stress-energy tensor of the brane, and $(p + 1)$-dimensional currents associated with its intrinsic angular momentum. These are the only free parameters that affect the brane dynamics in the pole-dipole approximation. In section 5, the examples of pointlike and stringlike matter are considered. In particular, the Nambu-Goto string is generalized to allow spinning matter on its ends. Section 6 is devoted to concluding remarks.

Our conventions are the same as in ref. [4], with the exception of the metric signature. Greek indices $\mu, \nu, \ldots$ are the spacetime indices, and run over $0, 1, \ldots, D-1$. Latin indices $a, b, \ldots$ are the world sheet indices and run over $0, 1, \ldots, p$. The Latin indices $i, j, \ldots$ refer to the world sheet boundary and take values $0, 1, \ldots, p - 1$. The coordinates of spacetime, world sheet and world sheet boundary are denoted by x^μ, ξ^a and λ^i, respectively. The corresponding metric tensors are denoted by $g_{\mu\nu}(x)$, $\gamma_{ab}(\xi)$ and $h_{ij}(\lambda)$. The signature convention is defined by $\mathrm{diag}(-, +, \ldots, +)$, and the indices are raised by the inverse metrics $g^{\mu\nu}$, γ^{ab} and h^{ij}.

2. Multipole expansion of the stress-energy tensor

A p-brane is an extended p-dimensional object whose trajectory is a $(p + 1)$-dimensional world sheet, commonly denoted by $\mathcal{M}$. In this paper, we shall be concerned with material objects shaped to resemble a p-brane. If this is the case, all but the first couple of terms in the multipole expansion around a suitably chosen $(p + 1)$-dimensional surface can be neglected. Retaining the first two terms defines the so called pole-dipole approximation.

Let us begin with the introduction of a $(p + 1)$-dimensional surface $x^\mu = z^\mu(\xi)$ in D-dimensional spacetime, where ξ^a are the surface coordinates. We shall assume that the surface is *everywhere regular*, and the coordinates ξ^a well defined. We shall consider only time infinite brane trajectories. This means that every spatial section of the spacetime has nonempty intersection with the world sheet. As for the intersection itself, it is supposed to be of finite length. Thus, only closed or finite open branes are considered. The world sheet boundary $\partial\mathcal{M}$ is parametrized by p coordinates λ^i.

In what follows, we shall frequently use the notion of the world sheet coordinate vectors

$$u_a^\mu = \frac{\partial z^\mu}{\partial \xi^a},$$

and the world sheet induced metric tensor

$$\gamma_{ab} = g_{\mu\nu} u_a^\mu u_b^\nu \,.$$

The induced metric is assumed to be nondegenerate, $\gamma \equiv \det(\gamma_{ab}) \neq 0$, and of Minkowski signature. With this assumption, each point on the world sheet accommodates a timelike tangent vector. This is how the notion of the timelike curve is generalized to a $(p+1)$-dimensional case.

Now, we are ready to expand the stress-energy tensor into a δ-function series around the surface $x^\mu = z^\mu(\xi)$. Generalizing the results of ref. [4], where this expansion has been used in the single-pole approximation, we define

$$T^{\mu\nu}(x) = \int d^{p+1}\xi \sqrt{-\gamma} \left[b^{\mu\nu}(\xi) \frac{\delta^{(4)}(x-z)}{\sqrt{-g}} + b^{\mu\nu\rho}(\xi) \nabla_\rho \frac{\delta^{(4)}(x-z)}{\sqrt{-g}} + \dots \right], \qquad (2.1)$$

where ∇_ρ stands for the Riemannian covariant derivative, and $g(x)$ is the determinant of the target space metric $g_{\mu\nu}(x)$.

The decomposition (2.1) is suitable for treating matter which is well localized around the brane $x^\mu = z^\mu(\xi)$. In fact, the stress-energy tensor $T^{\mu\nu}(x)$ must drop exponentially to zero as we move away from the brane if we want the series (2.1) to be well defined. If this is the case, each coefficient $b^{\mu\nu\rho_1\dots\rho_n}$ gets smaller as n gets larger. In the lowest, single-pole approximation, all b's except the first are neglected, and we end up with the manifestly covariant expression analyzed in ref. [4]. In this paper, we extend the analysis to the pole-dipole approximation, defined by neglecting all but the first two b-coefficients.

2.1 Diffeomorphism invariance

The series (2.1) can, in general, be truncated at any level. As opposed to the single-pole approximation, however, the general truncation turns out not to be manifestly covariant. Indeed, the transformation properties of the b-coefficients, as derived from the known transformation law of the stress-energy tensor $T^{\mu\nu}$, show that b's are not tensors. This leaves us with two tasks to be accomplished. The first is to show that the general truncation of the series (2.1) is diffeomorphism invariant. The second is to find a manifestly covariant form of the truncated expression.

Let us start with transformation properties of the b-coefficients. First, we define the scalar functional

$$T[f] \equiv \int d^D x \sqrt{-g}\, T^{\mu\nu}(x) f_{\mu\nu}(x)\,, \qquad (2.2)$$

where $f_{\mu\nu}(x)$ is an arbitrary tensor field with compact support. The decomposition (2.1) then yields

$$T[f] = \int d^{p+1}\xi \sqrt{-\gamma} \Big[I_0\left(b_0 f\right) + I_1\left(b_1 f\right) + \dots \Big], \qquad (2.3)$$

where

$$I_n\left(b_n f\right) = \int d^D x \sqrt{-g}\, b^{\mu\nu\rho_1\dots\rho_n} f_{\mu\nu} \nabla_{\rho_1} \dots \nabla_{\rho_n} \frac{\delta^{(4)}(x-z)}{\sqrt{-g}}\,. \qquad (2.4)$$

We can now make use of the compact support of the arbitrary functions $f_{\mu\nu}(x)$ to perform a series of partial integrations in (2.4). This leads to

$$I_n\left(b_n f\right) = (-1)^n \nabla_{\rho_n} \ldots \nabla_{\rho_1} b^{\mu\nu\rho_1\ldots\rho_n} f_{\mu\nu}\Big|_{x=z}. \tag{2.5}$$

In this expression, the action of the covariant derivative ∇_ρ on the b-coefficients is defined formally, by treating b's as tensors with no x dependence. Thus, the expression ∇b_n contains only Γb_n terms in accordance with the index structure of the b_n coefficient (e.g. $\nabla_\rho b^{\mu\nu}(\xi) = \Gamma^\mu{}_{\lambda\rho}(x) b^{\lambda\nu}(\xi) + \Gamma^\nu{}_{\lambda\rho}(x) b^{\mu\lambda}(\xi)$). Now, we perform differentiations in (2.5). The result can symbolically be written as

$$
\begin{aligned}
I_0(b_0 f) &= b_0 f(z), \\
I_1(b_1 f) &= -\left[(\nabla b_1)f + b_1 \nabla f\right]_{x=z}, \\
I_2(b_2 f) &= \left[(\nabla^2 b_2)f + 2(\nabla b_2)\nabla f + b_2 \nabla^2 f\right]_{x=z}, \ldots
\end{aligned}
$$

Using this in the decomposition (2.3), we can regroup the additive terms to obtain

$$T[f] = \int d^{p+1}\xi \sqrt{-\gamma}\Big[B^{\mu\nu} f_{\mu\nu}(z) + B^{\mu\nu\rho} f_{\mu\nu;\rho}(z) + B^{\mu\nu\rho\lambda} f_{\mu\nu;\rho\lambda}(z) + \ldots\Big]. \tag{2.6}$$

Here, $f_{\mu\nu;\rho_1\ldots\rho_n}(z)$ stands for $\nabla_{\rho_n}\ldots\nabla_{\rho_1} f_{\mu\nu}$ evaluated at $x = z(\xi)$, and the coefficients $B(\xi)$ have the general structure

$$
\begin{aligned}
B_0 &= b_0 - \nabla b_1 + \nabla^2 b_2 - \nabla^3 b_3 + \ldots, \\
B_1 &= \quad\;\; - b_1 + 2\nabla b_2 - 3\nabla^2 b_3 + \ldots, \\
B_2 &= \qquad\qquad\;\; b_2 - 3\nabla b_3 + \ldots, \ldots
\end{aligned} \tag{2.7a}
$$

We see that the system of equations (2.7a) can be solved for b's. Symbolically,

$$
\begin{aligned}
b_0 &= B_0 - \nabla B_1 + \nabla^2 B_2 - \nabla^3 B_3 + \ldots, \\
b_1 &= \quad\;\; - B_1 + 2\nabla B_2 - 3\nabla^2 B_3 + \ldots, \\
b_2 &= \qquad\qquad\;\; B_2 - 3\nabla B_3 + \ldots, \ldots
\end{aligned} \tag{2.7b}
$$

The obtained results lead us to two important conclusions. First,

- B-coefficients are tensors with respect to spacetime diffeomorphisms.

This is a consequence of the fact that $T[f]$ in (2.6) is a scalar functional for any choice of the tensor field $f_{\mu\nu}(x)$. The corresponding transformation law reads

$$B'^{\mu_1\ldots\mu_n} = \left(\frac{\partial x'^{\mu_1}}{\partial x^{\nu_1}} \cdots \frac{\partial x'^{\mu_n}}{\partial x^{\nu_n}}\right)_{x=z} B^{\nu_1\ldots\nu_n}. \tag{2.8}$$

The transformation properties of the b-coefficients are derived from (2.7b), and do not have tensorial character. Second,

- truncation of the series (2.1) at any level is a covariant operation.

Indeed, if all the b's of the order n and higher are put to zero ($b_n = b_{n+1} = \cdots = 0$), the corresponding B's will also vanish ($B_n = B_{n+1} = \cdots = 0$), as is seen from (2.7a). Being tensors, the zero B's will remain to be zero in any reference frame ($B'_n = B'_{n+1} = \cdots = 0$), and according to (2.7b) so will the corresponding b's ($b'_n = b'_{n+1} = \cdots = 0$). Thus, the truncation is diffeomorphism invariant.

Let us consider two simple examples. The *single-pole* approximation is defined by retaining only the leading term b_0, while $b_1 = b_2 = \cdots = 0$. The equations (2.7a) then give

$$b^{\mu\nu} = B^{\mu\nu} \,,$$

which means that $b^{\mu\nu}$, in the single-pole approximation, transforms as a tensor. The stress-energy tensor has a manifestly covariant form

$$T^{\mu\nu}(x) = \int d^{p+1}\xi \sqrt{-\gamma} b^{\mu\nu} \frac{\delta^{(D)}(x-z)}{\sqrt{-g}} \,.$$

In the *pole-dipole* approximation, the first two coefficients b_0 and b_1 are retained, while the remaining $b_2, b_3, \ldots$ are put to zero. The system of equations (2.7a) reduces to

$$B^{\mu\nu} = b^{\mu\nu} - \nabla_\rho b^{\mu\nu\rho}, \qquad B^{\mu\nu\rho} = -b^{\mu\nu\rho} \,, \tag{2.9}$$

where $\nabla_\rho b^{\mu\nu\rho} \equiv \Gamma^\mu{}_{\lambda\rho} b^{\lambda\nu\rho} + \Gamma^\nu{}_{\lambda\rho} b^{\mu\lambda\rho} + \Gamma^\rho{}_{\lambda\rho} b^{\mu\nu\lambda}$. We see that the coefficient $b^{\mu\nu\rho}$ transforms as a tensor in this approximation, while $b^{\mu\nu}$ does not. The stress-energy tensor is rewritten in a manifestly covariant form

$$T^{\mu\nu}(x) = \int d^{p+1}\xi \sqrt{-\gamma} \left[B^{\mu\nu} \frac{\delta^{(D)}(x-z)}{\sqrt{-g}} - \nabla_\rho \left(B^{\mu\nu\rho} \frac{\delta^{(D)}(x-z)}{\sqrt{-g}} \right) \right] \,. \tag{2.10}$$

In this form, the decomposition of the stress-energy tensor is manifestly covariant with respect to both, spacetime diffeomorphisms and world sheet reparametrizations. The corresponding transformation properties are summarized as follows:

	$B^{\mu\nu}(\xi)$	$B^{\mu\nu\rho}(\xi)$	$\gamma_{ab}(\xi)$	$g_{\mu\nu}(x)$
spacetime	tensor	tensor	scalar	tensor
world sheet	scalar	scalar	tensor	scalar

2.2 Extra symmetry 1

In this subsection, we shall demonstrate the appearance of an additional gauge transformation that leaves the stress-energy tensor invariant. To this end, note that each term in the decomposition (2.1) basically contains $D - p - 1$ δ-functions, which are used to model a p-brane in a D-dimensional spacetime. The extra $p + 1$ δ-functions and extra $p + 1$ integrations are introduced only to covariantize the expressions. This observation leads us to conclude that there are redundant b-coefficients in (2.1). In particular, the derivatives parallel to the world sheet are integrated out, as they should, considering the fact that matter is not localized along the brane. As a consequence, the parallel components of $b^{\mu\nu\rho}$

coefficients are expected to dissapear from the decomposition (2.1). To check this, let us define the transformation law of the form

$$\delta_1 b^{\mu\nu\rho} = \epsilon^{\mu\nu a} u_a^\rho \,, \tag{2.11a}$$

where $\epsilon^{\mu\nu a}(\xi) = \epsilon^{\nu\mu a}(\xi)$ are free parameters. Using (2.11a) to calculate the variation of the functional $T[f]$, we find that the invariance of the stress-energy tensor requires an additional transformation of the $B^{\mu\nu}$ coefficients. Precisely,

$$\delta_1 B^{\mu\nu} = -\nabla_a \epsilon^{\mu\nu a} \,, \tag{2.11b}$$

where ∇_a stands for the total covariant derivative, defined in the appendix. In fact, the transformation law (2.11) defines a symmetry of the stress-energy tensor only if the boundary terms are missing. Indeed, the variation of the functional $T[f]$ under (2.11) has the form

$$T'[f] = T[f] - \int_{\partial \mathcal{M}} d^p \lambda \sqrt{-h} \, n_a \epsilon^{\mu\nu a} f_{\mu\nu} \,,$$

where $h_{ij}(\lambda)$ is the induced metric on $\partial \mathcal{M}$, and $n^a(\lambda)$ is the unit boundary normal (see the appendix). To have the full invariance, the parameters $\epsilon^{\mu\nu a}$ are required to obey the boundary conditions

$$n_a \epsilon^{\mu\nu a}\Big|_{\partial \mathcal{M}} = 0 \,. \tag{2.12}$$

The transformation rule (2.11), with parameters constrained by (2.12), defines the *extra symmetry* 1 of the brane dynamics.

Now we see that parallel components of the $b^{\mu\nu\rho}$ coefficients are indeed pure gauge,

$$\delta_1 (b^{\mu\nu\rho} u_\rho^a) = \epsilon^{\mu\nu a} \,.$$

They can be gauged away everywhere except on the boundary, where the parameters $\epsilon^{\mu\nu a}$ are not free. As a consequence, the theory will contain some peculiar degrees of freedom, which live exclusively on the boundary, and do not appear in the world sheet equations. In the next section, we shall clarify their physical meaning.

2.3 Extra symmetry 2

The expansion of the stress-energy tensor into a δ-function series (2.1) has been performed with an arbitrary choice of the surface $x^\mu = z^\mu(\xi)$. If we use another surface, let us say $x^\mu = z'^\mu(\xi)$, the coefficients $b^{\mu\nu}$, $b^{\mu\nu\rho}$, ... will change to $b'^{\mu\nu}$, $b'^{\mu\nu\rho}$, ..., while leaving the stress-energy tensor invariant. The transformation law of the b-coefficients, generated by the replacement $z^\mu \to z'^\mu$, defines the gauge symmetry that we shall call *extra symmetry* 2.

The extra symmetry 2 is an exact symmetry of the full expansion (2.1), but only approximate symmetry of the truncated series (2.10). This is because the condition $b_2 = b_3 = \cdots = 0$ is a gauge condition that fixes the choice of the surface $x^\mu = z^\mu(\xi)$. Indeed, if the surface is chosen to lie outside the region where matter is localized, the higher b's will give a substantial contribution to the series, no matter how thin the brane is. The best we

can do is to keep the surface $x^\mu = z^\mu(\xi)$ inside the localized matter. Then, we can assume the following hierarchy of the b-coefficients:

$$b^{\mu\nu} = \mathcal{O}_0, \quad b^{\mu\nu\rho} = \mathcal{O}_1, \quad b^{\mu\nu\rho\lambda} = \mathcal{O}_2, \ \ldots,$$

where $\mathcal{O}_n$ stands for the order of smallness. The truncation of the n-th order is defined as an approximation in which the $\mathcal{O}_{n+1}$ and higher terms are neglected. In this approximation, the parameters of the extra symmetry 2, as defined by

$$z'^\mu(\xi) = z^\mu(\xi) + \epsilon^\mu(\xi), \tag{2.13}$$

are constrained by the requirement

$$b'_{n+1} = \mathcal{O}_{n+1}. \tag{2.14}$$

The transformation law (2.13) generates the corresponding transformation of the b-coefficients. It is shown to have the general form

$$
\begin{aligned}
b'_0 &= b_0 + b_0\epsilon + b_1\epsilon + \ldots, \\
b'_1 &= b_1 + b_0\epsilon + b_1\epsilon + \ldots, \\
b'_2 &= b_2 + b_1\epsilon + b_2\epsilon + \ldots, \ \ldots
\end{aligned}
\tag{2.15}
$$

In both, single-pole and pole-dipole approximations, the transformation (2.15) and the constraint (2.14) imply

$$\epsilon^\mu = \mathcal{O}_1. \tag{2.16}$$

This condition ensures that the order of truncation is not spoiled by the action of extra symmetry 2.

The transformation rule (2.15) can be rewritten in terms of B-coefficients. Discarding contributions of the order $\mathcal{O}_2$ and higher, we obtain

$$
\begin{aligned}
\delta_2 B^{\mu\nu} &= -B^{\mu\nu} u_\rho^a \nabla_a \epsilon^\rho - 2B^{\lambda(\mu} \varGamma^{\nu)}{}_{\lambda\rho} \epsilon^\rho, \\
\delta_2 B^{\mu\nu\rho} &= -B^{\mu\nu} \epsilon^\rho.
\end{aligned}
\tag{2.17}
$$

The equations (2.17) and (2.13) define the extra symmetry 2 in the pole-dipole approximation.

Three remarks are in order. First, notice that the parameter ϵ^μ, as defined by (2.13), is a spacetime vector. Then, the explicit presence of the connection in the transformation law (2.17) seems to contradict the tensorial character of the B-coefficients. In fact, there is no contradiction. The transformation law of the B-coefficients under spacetime diffeomorphisms is given by (2.8). We see that all x-dependent terms are evaluated on the surface $x^\mu = z^\mu(\xi)$. When the surface is changed by (2.13), the new coefficients are given by (2.17). Their transformation law under diffeomorphisms is shown to have the same form as in (2.8), the only difference being that the x-dependent terms are now evaluated on the new surface, $x^\mu = z'^\mu(\xi)$.

The second remark concerns the single-pole approximation. We have seen that the invariance of every truncation implies the constraint $\epsilon^\mu = \mathcal{O}_1$. Using this in the single-pole

approximation, which is defined by neglecting $\mathcal{O}_1$ terms, we obtain $\delta_2 z^\mu = 0$, $\delta_2 B^{\mu\nu} = 0$. Thus, the extra symmetry 2 in the single-pole approximation is trivial. This is a consequence of the fact that single-pole branes are infinitely thin, which leaves no freedom for the choice of $z^\mu(\xi)$.

Finally, let us observe that fixing the gauge of extra symmetry 2 defines what could be called *central surface of mass* distribution for our localized matter. In the particle case (0-brane), this coincides with the usual notion of the centre of mass. We shall see in section 4 how a proper definition of the central surface of mass simplifies the world sheet equations, and helps us to interpret free parameters of the theory.

3. World sheet equations

In this section we shall analyze the stress-energy conservation equation (1.1) in the pole-dipole approximation. We first define an arbitrary vector field $f_\mu(x)$ of *compact support*, and rewrite the equation (1.1) in the convenient form

$$\int d^D x \sqrt{-g} f_\mu \nabla_\nu T^{\mu\nu} = 0, \qquad \forall f_\mu(x). \tag{3.1}$$

Now, we use the decomposition (2.10) of the stress-energy tensor. Owing to the compact support of $f_\mu(x)$, we are allowed to change the order of integrations, and to drop surface terms. Thus, we arrive at

$$\int d^{p+1}\xi \sqrt{-\gamma} \left(B^{\mu\nu} f_{\mu;\nu} + B^{\mu\nu\rho} f_{\mu;\nu\rho} \right) = 0, \tag{3.2}$$

where $f_{\mu;\nu} \equiv (\nabla_\nu f_\mu)_{x=z}$, $f_{\mu;\nu\rho} \equiv (\nabla_\rho \nabla_\nu f_\mu)_{x=z}$. The fact that this equation holds for every $f_\mu(x)$ puts some constraints on the coefficients $B^{\mu\nu}$ and $B^{\mu\nu\rho}$. To find these, we decompose the derivatives of the vector field $f_\mu(x)$ into components orthogonal and parallel to the surface $x^\mu = z^\mu(\xi)$:

$$f_{\mu;\lambda} = f_{\mu\lambda}^\perp + u_\lambda^a \nabla_a f_\mu, \tag{3.3a}$$

$$f_{\mu;(\lambda\rho)} = f_{\mu\lambda\rho}^\perp + 2f_{\mu(\lambda a}^\perp u_{\rho)}^a + f_{\mu ab} u_\lambda^a u_\rho^b, \tag{3.3b}$$

$$f_{\mu;[\lambda\rho]} = \frac{1}{2} R^\sigma{}_{\mu\lambda\rho} f_\sigma. \tag{3.3c}$$

Here, the orthogonal and parallel components are obtained by using the projectors

$$P_\perp{}^\mu_\nu = \delta^\mu_\nu - u_a^\mu u_\nu^a, \qquad P_\parallel{}^\mu_\nu = u_a^\mu u_\nu^a. \tag{3.4}$$

More precisely, $f_{\mu\lambda}^\perp = P_\perp{}^\sigma_\lambda f_{\mu;\sigma}$, $f_{\mu\lambda\rho}^\perp = P_\perp{}^\sigma_\lambda P_\perp{}^\nu_\rho f_{\mu;\sigma\nu}$, $f_{\mu\lambda a}^\perp = P_\perp{}^\sigma_\lambda u_a^\nu f_{\mu;(\sigma\nu)}$ and $f_{\mu ab} = u_a^\sigma u_b^\nu f_{\mu;(\sigma\nu)}$. Direct calculation yields

$$\begin{aligned} f_{\mu ab} &= \nabla_{(a}\nabla_{b)} f_\mu - (\nabla_a u_b^\nu) f_{\mu\nu}^\perp, \\ f_{\mu\rho a}^\perp &= P_\perp{}^\nu_\rho \nabla_a f_{\mu\nu}^\perp + (\nabla_a u_\rho^b) \nabla_b f_\mu + \frac{1}{2} P_\perp{}^\lambda_\rho u_a^\nu R^\sigma{}_{\mu\nu\lambda} f_\sigma, \end{aligned} \tag{3.5}$$

which tells us that the only independent components on the surface $x^\mu = z^\mu(\xi)$ are f_μ, $f_{\mu\nu}^\perp$ and $f_{\mu\nu\rho}^\perp$. We can now use (3.3) and (3.5) in the equations (3.2) to group the coefficients

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into terms proportional to the independent derivatives of f_μ. The obtained equation has the following general structure:

$$\int_{\mathcal{M}} d^{p+1}\xi \sqrt{-\gamma} \Big[X^{\mu\nu\rho} f^\perp_{\mu\nu\rho} + X^{\mu\nu} f^\perp_{\mu\nu} + X^\mu f_\mu + \nabla_a \left(X^{\mu\nu a} f^\perp_{\mu\nu} + X^{\mu ab} \nabla_b f_\mu + X^{\mu a} f_\mu \right) \Big] = 0\,.$$

Owing to the fact that f_μ, $f^\perp_{\mu\nu}$ and $f^\perp_{\mu\nu\rho}$ are independent functions on the world sheet, we deduce that the first three terms must separately vanish. The resulting equations read:

$$P_{\perp\lambda}^{\ \nu} P_{\perp\rho}^{\ \sigma} B^{\mu(\lambda\rho)} = 0\,, \quad (3.6a)$$

$$P_{\perp\nu}^{\ \sigma} \Big[B^{\mu\nu} - \nabla_a \left(B^{\mu\rho\nu} + P_{\perp\lambda}^{\ \nu} B^{\mu\lambda\rho} \right) \Big] = 0\,, \quad (3.6b)$$

$$\nabla_b \Big(B^{\mu\nu} u^b_\nu + 2 B^{\mu(\lambda\rho)} u^a_\lambda \nabla_a u^b_\rho - \nabla_a B^{\mu(\lambda\rho)} u^a_\lambda u^b_\rho \Big) - \left(P_{\perp\sigma}^{\ \rho} B^{\nu(\lambda\sigma)} + \frac{1}{2} B^{\nu\lambda\rho} \right) R^\mu_{\ \nu\lambda\rho} = 0 \quad (3.6c)$$

This leaves us with the surface integral that vanishes itself:

$$\int_{\partial\mathcal{M}} d^p\lambda \sqrt{-h}\, n_a \left(X^{\mu\nu a} f^\perp_{\mu\nu} + X^{\mu ab} \nabla_b f_\mu + X^{\mu a} f_\mu \right) = 0\,. \qquad (3.7)$$

The components $f^\perp_{\mu\nu}$ and f_μ, when evaluated on the boundary, are mutually independent, but $\nabla_a f_\mu$ is not. This is why we decompose the ∇_a derivative into components orthogonal and parallel to the boundary:

$$\nabla_a f_\mu = n_a \nabla_\perp f_\mu + v^i_a \nabla_i f_\mu\,. \qquad (3.8)$$

Here, $\nabla_\perp \equiv n^a \nabla_a$, ∇_i is the total covariant derivative on $\partial\mathcal{M}$, and v^a_i are the boundary coordinate vectors (see the appendix for details). Now, $f^\perp_{\mu\nu}$, $\nabla_\perp f_\mu$ and f_μ are mutually independent, and the equation (3.7) yields three sets of boundary conditions:

$$P_{\perp\lambda}^{\ \nu} B^{\mu(\lambda\rho)} u^a_\rho n_a \Big|_{\partial\mathcal{M}} = 0\,, \quad (3.9a)$$

$$B^{\mu\lambda\rho} u^a_\lambda u^b_\rho n_a n_b \Big|_{\partial\mathcal{M}} = 0\,, \quad (3.9b)$$

$$\Big[\nabla_i \left(B^{\mu(\lambda\rho)} u^a_\lambda u^b_\rho v^i_a n_b \right) - n_b \left(B^{\mu\nu} u^b_\nu + 2 B^{\mu(\lambda\rho)} u^a_\lambda \nabla_a u^b_\rho - \nabla_a B^{\mu(\lambda\rho)} u^a_\lambda u^b_\rho \right) \Big] \Big|_{\partial\mathcal{M}} = 0 \quad (3.9c)$$

The equations (3.6) and (3.9) describe branelike matter in the pole-dipole approximation. As we can see, the basic variables z^μ, $B^{\mu\nu}$ and $B^{\mu\nu\rho}$ are mixed in a way that makes it difficult to recognize their physical meaning. In what follows, we shall decompose the B-coefficients into components orthogonal and parallel to the world sheet, and try to diagonalize the world sheet equations.

We begin with the $B^{\mu\nu\rho}$ coefficients. Using the constraint (3.6a) to eliminate some orthogonal components, we arrive at

$$B^{\mu\nu\rho} = 2 u^{(\mu}_b B^{\nu)\rho b}_\perp + u^\mu_a u^\nu_b B^{\rho ab}_\perp + u^\rho_a B^{\mu\nu a}\,, \qquad (3.10)$$

where $B^{(\mu\nu)a}_\perp \equiv B^{\mu[ab]}_\perp \equiv B^{[\mu\nu]a} \equiv 0$. Note that the $B^{\mu\nu a}$ component is left as is, neither orthogonal nor parallel to the world sheet. This is because we remember the extra symmetry 1, which tells us that $B^{\mu\nu a}$ is likely to drop from the diagonalized world sheet equations. Now, we use (3.10) and rewrite equation (3.6b) in the form

$$P_{\perp\nu}^{\ \rho} \Big[B^{\mu\nu} - \nabla_a \left(S^{\mu\nu a} + N^{\mu\nu a} \right) \Big] = 0\,, \qquad (3.11)$$

where

$$S^{\mu\nu a} \equiv B_\perp^{\mu\nu a} + u_b^{[\mu} B_\perp^{\nu]ba}, \qquad N^{\mu\nu a} \equiv B^{\mu\nu a} + u_b^{(\mu} B_\perp^{\nu)ba}. \tag{3.12}$$

The new coefficients $S^{\mu\nu a}$ and $N^{\mu\nu a}$ are introduced for later convenience, and are neither orthogonal nor parallel to the world sheet. Instead, the defining relations (3.12) imply the constraint

$$S^{\mu\nu[a} u_\nu^{b]} = 0. \tag{3.13}$$

The coefficients $N^{\mu\nu a} = N^{\nu\mu a}$ and $S^{\mu\nu a} = -S^{\nu\mu a}$, subject to constraint (3.13), are in $1-1$ correspondence with $B^{\mu\nu\rho}$. In what follows, we shall rewrite the $B^{\mu\nu\rho}$ coefficients in all our equations in terms of $S^{\mu\nu a}$ and $N^{\mu\nu a}$:

$$B^{\mu\nu\rho} = 2u_a^{(\mu} S^{\nu)\rho a} + N^{\mu\nu a} u_a^\rho. \tag{3.14}$$

Let us now decompose the $B^{\mu\nu}$ coefficients. With the help of the projectors (3.4), we obtain

$$B^{\mu\nu} = B_\perp^{\mu\nu} + 2u_b^{(\mu} B_\perp^{\nu)b} + u_a^\mu u_b^\nu B^{ab}. \tag{3.15a}$$

When used in the equation (3.11), this decomposition yields

$$B_\perp^{\mu\nu} = P_{\perp\lambda}{}^\mu P_{\perp\rho}{}^\nu \nabla_a N^{\lambda\rho a}, \qquad B_\perp^{\mu a} = u_\lambda^a P_{\perp\rho}{}^\mu \nabla_b \left(S^{\lambda\rho b} + N^{\lambda\rho b} \right), \tag{3.15b}$$

and

$$P_{\perp\lambda}{}^\mu P_{\perp\rho}{}^\nu \nabla_a S^{\lambda\rho a} = 0. \tag{3.16a}$$

The equations (3.15b) and (3.16a) are equivalent to (3.11). The first shows that $B_\perp^{\mu\nu}$ and $B_\perp^{\mu a}$ are fully fixed by S and N. This leaves us with B^{ab}, $S^{\mu\nu a}$ and $N^{\mu\nu a}$ as the only independent coefficients in the theory. The second is viewed as a partial covariant conservation equation of the world sheet currents $S^{\mu\nu a}$.

Now, we can use (3.14) and (3.15) to rewrite the remaining equation (3.6c) in terms of the independent coefficients. By doing so, we arrive at

$$\nabla_b \left(m^{ab} u_a^\mu - 2u_\lambda^b \nabla_a S^{\mu\lambda a} + u_c^\mu u_\rho^c u_\lambda^b \nabla_a S^{\rho\lambda a} \right) - u_a^\nu S^{\lambda\rho a} R^\mu{}_{\nu\lambda\rho} = 0, \tag{3.16b}$$

where

$$m^{ab} \equiv B^{ab} - u_\rho^a u_\lambda^b \nabla_c N^{\rho\lambda c}. \tag{3.17}$$

The world sheet tensor m^{ab} is symmetric, and is used instead of B^{ab} in the set of free coefficients. As we can see, the coefficients $N^{\mu\nu a}$ have dropped from the world sheet equations (3.16), as expected. The physical meaning of the remaining coefficients, m^{ab} and $S^{\mu\nu a}$, will be clarified in the next section.

We can now apply the above procedure to the boundary conditions (3.9). Using the algebraic constraints (3.14), (3.15) and (3.17), the boundary conditions are rewritten in terms of the independent coefficients:

$$S^{\mu\nu a} n_a n_\nu \Big|_{\partial\mathcal{M}} = 0, \tag{3.18a}$$

$$P_{\perp\lambda}{}^\mu P_{\perp\rho}{}^\nu S^{\lambda\rho a} n_a \Big|_{\partial\mathcal{M}} = 0, \tag{3.18b}$$

$$\left[\nabla_i \left(N^{ij} v_j^\mu + 2S^{\mu\nu a} n_a v_\nu^i \right) - n_b \left(m^{ba} u_a^\mu - 2u_\nu^b \nabla_a S^{\mu\nu a} + u_c^\mu u_\rho^c u_\lambda^b \nabla_a S^{\rho\lambda a} \right) \right] \Big|_{\partial\mathcal{M}} = 0 \tag{3.18c}$$

where

$$N^{ij} \equiv N^{\mu\nu a} n_a v_\mu^i v_\nu^j. \tag{3.19}$$

The N^{ij} coefficients are defined on the boundary, and appear nowhere else.

The equations (3.16) and (3.18) are the main result of this paper. They are an equivalent of the covariant conservation equation (1.1) in the pole-dipole approximation, and determine the evolution of the brane. The free coefficients m^{ab}, $S^{\mu\nu a}$ and N^{ij} carry the information on the internal structure of the brane. In what follows, we shall analyze the physical meaning of these coefficients, and provide some examples.

4. Physical interpretation

The free coefficients m^{ab}, $S^{\mu\nu a}$ and N^{ij} characterize the internal structure of the brane. In this section, we shall analyze their physical meaning and transformation properties.

4.1 Symmetries

Let us first derive transformation properties of the free coefficients m^{ab}, $S^{\mu\nu a}$ and N^{ij}. To this end, we invert the decomposition equations (3.10), (3.15a), and rewrite the defining relations (3.12), (3.17) in terms of the original B-coefficients. The transformation properties of the B-coefficients have already been considered in section 2. It has been shown that B's are tensors with respect to both, spacetime and world sheet diffeomorphisms. As a consequence,

- m^{ab}, $S^{\mu\nu a}$ and N^{ij} are tensors of the type defined by their index structure.

In particular, N^{ij} is a second rank tensor with respect to the boundary reparametrizations.

The physical meaning of the m^{ab} coefficients is already known from the single-pole approximation [4]. It has been shown that m^{ab} represents the covariantly conserved effective $(p + 1)$-dimensional stress-energy tensor of the brane. In the pole-dipole approximation, its conservation is violated by the higher order terms.

In addition to diffeomorphisms, two extra symmetries have been discovered in section 2. The *extra symmetry* 1 is of the algebraic type, which ensures that only gauge invariant coefficients appear in a properly diagonalized world sheet equations. Indeed, the transformation laws (2.11) with the constraint (2.12) straightforwardly lead to:

$$\delta_1 m^{ab} = 0, \qquad \delta_1 S^{\mu\nu a} = 0, \qquad \delta_1 N^{ij} = 0. \tag{4.1}$$

The appearance of the peculiar N^{ij} coefficients that live exclusively on the boundary is a consequence of the constraint (2.12) that parameters of the extra symmetry 1 obey on the boundary. If not for this, the transformation law $\delta_1 N^{\mu\nu a} = -\epsilon^{\mu\nu a}$ would imply that $N^{\mu\nu a}$ are pure gauge *everywhere*, and would have to disappear from the gauge invariant world sheet equations. Physically, the N^{ij} coefficients characterize the tangential component of the brane thickness. Namely, when an infinitely thin brane is thickened, this is done in all spatial directions. Obviously, thickening in the directions tangential to the brane surface changes nothing in the brane interior. This is because matter is not localized along these

directions anyway. However, if the brane is open, the tangential thickening does influence the brane boundary. The boundary structure thus obtained is characterized by the N^{ij} coefficients. In fact, N^{ij} is a correction to the effective p-dimensional stress-energy tensor of the brane boundary, very much like m^{ab} is $(p+1)$-dimensional effective stress-energy tensor of the brane itself. The best way to see this is to consider a brane with extra massive matter attached to its boundary. The procedure of section 3 then yields the generalized boundary conditions in which the N^{ij} term appears as a correction to the effective boundary stress-energy tensor m^{ij}. An example of the kind is considered in the next section. It consists of the spinless string with massive, spinning particles attached to its ends.

The *extra symmetry* 2 has been defined in section 2 as the symmetry generated by the change of the surface $x^\mu = z^\mu(\xi)$ used in the δ-function expansion (2.1). The transformation laws (2.13), (2.17), thus obtained, can be used in the derivation of the corresponding transformation properties of the coefficients m^{ab}, $S^{\mu\nu a}$ and N^{ij}. We shall first decompose the parameters ϵ^μ into components orthogonal and parallel to the world sheet:

$$\epsilon^\mu = \epsilon_\perp^\mu + u_a^\mu \epsilon^a \,. \tag{4.2}$$

Then, the direct calculation yields

$$\delta_2 m^{ab} = -\left(u_\mu^c m^{ab} + u_\mu^{(a} m^{b)c}\right)\nabla_c \epsilon_\perp^\mu + \left(\epsilon^c \nabla_c m^{ab} - m^{bc}\nabla_c \epsilon^a - m^{ac}\nabla_c \epsilon^b\right)\,, \tag{4.3a}$$

$$\delta_2 S^{\mu\nu a} = -m^{ab} u_b^{[\mu} \epsilon_\perp^{\nu]} \,, \tag{4.3b}$$

$$\delta_2 N^{ij} = -m^{ab} v_a^i v_b^j \epsilon^c n_c \,, \tag{4.3c}$$

and, of course,

$$\delta_2 z^\mu = \epsilon_\perp^\mu + u_a^\mu \epsilon^a \,. \tag{4.3d}$$

This transformation rule leaves the world sheet equations (3.16) and (3.18) invariant. Notice, however, that the tangential parameters ϵ^a do not define a fully independent symmetry. This is because the subgroup defined by $\epsilon_\perp^\mu = 0$ and $\epsilon^a n_a|_{\partial\mathcal{M}} = 0$ coincides with the world sheet reparametrizations $\xi^{a\prime} = \xi^a + \epsilon^a(\xi)$. This is easily seen if we remember that $S^{\mu\nu a}$, N^{ij} and ϵ^μ are of the order $\mathcal{O}_1$, and that $\mathcal{O}_2$ terms are ignored in the pole-dipole approximation. The parameters ϵ^a which do not satisfy the boundary condition $\epsilon^a n_a|_{\partial\mathcal{M}} = 0$ cannot be associated with reparametrizations. This is why, in general, we cannot get rid of the ϵ^a part of the extra symmetry 2.

The transformation laws (4.3) are used for fixing the gauge freedom of the world sheet equations. As explained in section 2, the gauge fixing of the extra symmetry 2 corresponds to the choice of the *central surface of mass*—the surface that approximates a branelike matter distribution. In the particle case, it coincides with the usual notion of the centre of mass. We shall see later how an appropriate gauge fixing ensures that particle trajectories in flat spacetimes coincide with straight lines.

4.2 Intrinsic angular momentum

There are several ways one can associate the $S^{\mu\nu a}$ coefficients with the intrinsic angular momentum of the brane. One is to compare the 0-brane equations (3.16) with the Papapetrou result for the particle trajectory in the pole-dipole approximation [3]. Another is

the direct calculation of the angular momentum tensor $M^{\mu\nu\rho} \equiv T^{\rho[\mu}x^{\nu]}$. In this section, however, we shall simply count the number of independent charges associated with the $(p+1)$-currents $S^{\mu\nu a}$.

Let us, first, choose an appropriate coordinate system. To this end, we pick an arbitrary point of the brane, and attach inertial spacetime and world sheet frames to it. With this, $g_{\mu\nu}$ and γ_{ab} in the chosen point reduce to $\eta_{\mu\nu}$ and η_{ab}, respectively. Then, an additional Lorentz rotation of the spacetime frame is performed to ensure its comoving character:

$$u_a^\mu = \delta_a^\mu \,.$$

In this gauge, the algebraic constraint (3.13) reduces to

$$S^{\mu ab} = S^{\mu ba} \,. \tag{4.4a}$$

Now, we count the number of independent charge densities $S^{\mu\nu 0}$. In the first step, we use the constraint (4.4a), and the antisymmetry condition

$$S^{\mu\nu a} = -S^{\nu\mu a} \tag{4.4b}$$

to rule out the vanishing S^{abc} coefficients. This leaves us with the charge densities $S^{\bar\mu\bar\nu 0}$ and $S^{\bar\mu a0}$. (Here, the index decomposition $\mu = (a, \bar\mu)$ is used.) As $\bar\mu$ takes $D-p-1$ values, there are $(D-p-1)(D-p-2)/2$ independent $S^{\bar\mu\bar\nu 0}$ coefficients, and $(D-p-1)(p+1)$ independent $S^{\bar\mu a0}$ coefficients. In total, there are

$$\frac{D(D-1)}{2} - \frac{(p+1)p}{2} \equiv \dim\left[\mathrm{SO}(1,D-1)\right] - \dim\left[\mathrm{SO}(1,p)\right] \tag{4.5a}$$

independent charge densities $S^{\mu\nu 0}$.

As we can see, the number of independent charges associated with the currents $S^{\mu\nu a}$ is given as a difference of two terms. The first coincides with the dimension of $\mathrm{SO}(1, D-1)$ group, or equivalently, the number of independent Lorentz rotations in D spacetime dimensions. The second is the dimension of $\mathrm{SO}(1,p)$ group, i.e. the number of independent Lorentz rotations in $(p+1)$-dimensional world sheet. Thus, our charges correspond to the Lorentz rotations perpendicular to the world sheet. Naturally, we associate them with the intrinsic angular momentum of the brane.

Notice that among the charges $S^{\mu\nu 0}$ there are none corresponding to the tangential world sheet rotations. This is because they are already taken into account through the effective stress-energy tensor m^{ab} of the brane. Indeed, these rotations do not require a nontrivial brane thickness — they exist already in the single-pole approximation. In contrast, the possibility to have perpendicular rotations in the *comoving frame* demands a thick brane, as simulated by the pole-dipole approximation. The $S^{\mu\nu a}$ coefficients "measure" the brane thickness, and have nothing to do with infinitely thin branes. As a consequence, the angular momentum components associated with the $p(p+1)/2$ tangential rotations are related to m^{ab} rather than $S^{\mu\nu a}$.

In what follows, we shall use the notation $a = (0, \bar a)$ to further decompose the $S^{\mu\nu a}$ coefficients. Thus, the nonvanishing charge densities are written as $S^{\bar\mu\bar\nu 0}$, $S^{\bar\mu\bar a 0}$ and $S^{\bar\mu 00}$.

They correspond to the $\bar{\mu} - \bar{\nu}$, $\bar{\mu} - \bar{a}$ and $\bar{\mu} - 0$ rotation planes, respectively. The $S^{\bar{\mu}\bar{\nu}0}$ and $S^{\bar{\mu}\bar{a}0}$ are the spatial angular momentum components, and $S^{\bar{\mu}00}$ are boosts.

Now, we can use the gauge freedom of extra symmetry 2 to fix some unphysical coefficients. To this end, the transformation law (4.3b) is rewritten in the comoving frame $u^\mu_a = \delta^\mu_a$, and applied to the boosts $S^{\bar{\mu}00}$. The resulting rule

$$\delta_2 S^{\bar{\mu}00} = \frac{1}{2} m^{00} \epsilon^{\bar{\mu}}$$

shows that the boosts $S^{\bar{\mu}00}$ are pure gauge, and can be gauged away. Thus, we are left with the spatial angular momentum densities $S^{\bar{\mu}\bar{\nu}0}$ and $S^{\bar{\mu}\bar{a}0}$ as the only physical charge densities associated with the currents $S^{\mu\nu a}$. By direct counting, we find that there are precisely

$$\frac{(D-1)(D-2)}{2} - \frac{p(p-1)}{2} \equiv \dim\left[\mathrm{SO}(D-1)\right] - \dim\left[\mathrm{SO}(p)\right] \qquad (4.5b)$$

independent charges. They correspond to the *spatial rotations perpendicular to the brane*. In what follows, the intrinsic angular momentum of the brane will be referred to as classical spin, or simply spin, for short. It should not be confused with the usual notion of spin, which originates from the nonvanishing spin-tensor.

By inspecting the world sheet equations (3.16), we see that the currents $S^{\mu\nu a}$ are coupled to both, the spacetime curvature, and the brane orbital degrees of freedom. It is only in the particle case that the spin-orbit interaction can be gauged away. This is done by the proper definition of the particle centre of mass. As a consequence, the particle trajectories in flat spacetime coincide with straight lines.

In what follows, we shall consider some examples to demonstrate the influence of classical spin on the brane dynamics and conserved quantities.

5. Examples

In this section, the $p = 0$ and $p = 1$ branes are considered in 4 spacetime dimensions. Let us first analyze the general particle case.

5.1 Particle

The world sheet of a particle is one-dimensional, and is called world line. We shall parametrize it with the proper distance s, thereby fixing the reparametrization invariance:

$$\gamma = u^\mu u_\mu = -1 \,.$$

Here, and in what follows, the indices $a, b, \dots$ are omitted, as they take only one value. Thus, the world line equations (3.16) are rewritten as

$$P_{\perp\lambda}^{\ \ \mu} P_{\perp\rho}^{\ \ \nu} \frac{DS^{\lambda\rho}}{ds} = 0 \,, \qquad (5.1a)$$

$$\frac{D}{ds}\left(mu^\mu - 2u_\nu \frac{DS^{\mu\nu}}{ds}\right) - u^\nu S^{\lambda\rho} R^\mu_{\ \nu\lambda\rho} = 0 \,, \qquad (5.1b)$$

where $Dv^\mu/ds \equiv dv^\mu/ds + \Gamma^\mu{}_{\lambda\rho} u^\lambda v^\rho$. These equations are the same as obtained by Papapetrou [3]. The coefficients $S^{\mu\nu}$ are antisymmetric, but otherwise arbitrary (the constraint (3.13) is identically satisfied in the $p = 0$ case). We can, still, use the gauge freedom of the extra symmetry 2 to fix the $S^{\mu\nu} u_\nu$ components. Indeed, the transformation law (4.3b) implies

$$\delta_2(S^{\mu\nu} u_\nu) = \frac{m}{2} \epsilon_\perp^\mu,$$

wherefrom we see that parallel components of $S^{\mu\nu}$ can be gauged away. This leaves us with

$$S^{\mu\nu} = S_\perp^{\mu\nu} \,. \tag{5.2}$$

The fact that $S_\perp^{\mu\nu}$ is orthogonal to the world line is used in the derivation of the conservation laws. First, we project (5.1b) onto u_μ, and obtain

$$\frac{Dm}{ds} = \frac{dm}{ds} = 0 \,. \tag{5.3}$$

Thus, the mass parameter m is conserved along the world line. As a consequence, the equation (5.1b) implies

$$\frac{Du^\mu}{ds} = \mathcal{O}_1 \,.$$

Using this, and the fact that $\mathcal{O}_2$ terms are discarded in the pole-dipole approximation, the world line equations (5.1) are rewritten as

$$\frac{DS_\perp^{\mu\nu}}{ds} = 0 \,, \tag{5.4a}$$

$$m \frac{Du^\mu}{ds} = R^\mu{}_{\nu\lambda\rho} S_\perp^{\lambda\rho} u^\nu \,. \tag{5.4b}$$

As we can see, the intrinsic angular momentum $S_\perp^{\mu\nu}$ is covariantly conserved, and measures geodesic deviation of the particle trajectory.

Finally, let us mention that the boundary conditions (3.18) are absent in the $p = 0$ case.

5.2 String

The string trajectory is a two-dimensional world sheet with one-dimensional boundary. As in the particle case, the boundary line will be parametrized with the proper distance s, and the indices $i, j, \ldots$, which take only one value, will be omitted. Thus, the boundary metric h, and the tangent vector v^a satisfy

$$h = v^a v_a = -1 \,.$$

The only peculiarity of the string dynamics, as compared to higher branes, is the possibility to gauge away the N^{ij} coefficients. Indeed, there is only one such component in the string case, and one free parameter in the transformation law (4.3c):

$$\delta_2 N = -m^{ab} v_a v_b \epsilon \,,$$

where $\epsilon \equiv \epsilon^a n_a$. Thus, one can fix the gauge $N = 0$, whereupon the parameters ϵ^a are constrained to obey $\epsilon^a n_a|_{\partial \mathcal{M}} = 0$. With this condition, the ϵ^a part of the extra symmetry 2 reduces to the reparametrizations.

In what follows, we shall describe two specific string configurations. The first is a massive rod rotating around its longitudinal axis. The second is a spinless Nambu-Goto string with massive spinning particles attached to its ends.

Spinning rod. In this example, a massive rod slowly spinning around its longitudinal axis is considered. For simplicity, we choose flat spacetime ($R^{\mu}{}_{\nu\lambda\rho} = 0$), and Cartesian coordinates ($g_{\mu\nu}(x) = \eta_{\mu\nu}$).

The simple solution we shall look for is described as follows. The rod is at rest, and lies along the x-axis between the points $x = L/2$ and $x = -L/2$. It rotates around its longitudinal axis, so that

$$S^a \equiv S^{23a} = -S^{32a}$$

are the only nonvanishing $S^{\mu\nu a}$ currents. The world sheet coordinates ξ^a are fixed by the reparametrization gauge $z^a(\xi) = \xi^a$, while the boundary parameter λ coincides with the proper length s. As a consequence,

$$u^{\mu}_a = \delta^{\mu}_a\,, \qquad v^a = \delta^a_0\,, \qquad \gamma_{ab} = \eta_{ab}\,, \qquad h = -1\,.$$

One easily verifies that this is a solution of the world sheet equations (3.16) and the boundary conditions (3.18), provided

$$\partial_a m^{ab} = 0\,, \qquad \partial_a S^a = 0\,, \tag{5.5a}$$

and

$$m^{a1}\left(\xi^1 = \pm L/2\right) = 0\,, \qquad S^1\left(\xi^1 = \pm L/2\right) = 0\,. \tag{5.5b}$$

The equations (5.5a) tell us that the effective stress-energy tensor m^{ab} and the angular momentum current S^a are conserved quantities. The equations (5.5b) state that there is no flow of energy, momentum and angular momentum through the boundary.

The only thing that might not be obvious in this example is that the rod is indeed spinning around its longitudinal axis. To check this, we calculate the total angular momentum

$$J^{\mu\nu} \equiv \int d^3x \; x^{[\mu} T^{\nu]0}\,, \tag{5.6}$$

and find

$$J^{23} = \int_{-\frac{L}{2}}^{\frac{L}{2}} dx S^0(t, x)\,, \qquad J^{12} = J^{13} = 0\,.$$

Thus, the rod is indeed rotating in the $y - z$ plane. At the same time, the energy of the rod, as given by

$$E = \int d^3x \; T^{00}\,, \tag{5.7}$$

is shown to coincide with the rod mass:

$$E = \int_{-\frac{L}{2}}^{\frac{L}{2}} dx \; m^{00}(t, x)\,.$$

The absence of the kinetic term due to rotation is a consequence of the adopted approximation. Indeed, the rotational energy is quadratic in S^a, which gives the negligible $\mathcal{O}_2$ contribution to the total energy.

Let us notice, in the end, that both the angular momentum $\vec{J}$ and the energy E are conserved during evolution. This follows immediately from the world sheet equations (5.5a) and boundary conditions (5.5b).

Generalized Nambu-Goto string. In this example, we shall consider a spinless Nambu-Goto string with massive spinning particles attached to its ends. The stress-energy tensor is written as a sum of two terms,

$$T^{\mu\nu} = T_{\mathrm{s}}^{\mu\nu} + T_{\mathrm{p}}^{\mu\nu} \,, \tag{5.8a}$$

where

$$T_{\mathrm{s}}^{\mu\nu} = \int_{\mathcal{M}} d^2\xi \sqrt{-\gamma} B_{\mathrm{s}}^{\mu\nu} \frac{\delta^{(4)}(x-z)}{\sqrt{-g}} \,, \tag{5.8b}$$

$$T_{\mathrm{p}}^{\mu\nu} = \int_{\partial\mathcal{M}} d\lambda \sqrt{-h} \left(B_{\mathrm{p}}^{\mu\nu} \frac{\delta^{(4)}(x-z)}{\sqrt{-g}} - \nabla_\rho B_{\mathrm{p}}^{\mu\nu\rho} \frac{\delta^{(4)}(x-z)}{\sqrt{-g}} \right) . \tag{5.8c}$$

The string part of the stress-energy tensor is written in the single-pole approximation, in accordance with the assumed absence of spin in the string interior. The usual procedure then yields the familiar world sheet equations

$$\nabla_a \left(m^{ab} u_b^\mu \right) = 0 \,. \tag{5.9}$$

The particle part $T_{\mathrm{p}}^{\mu\nu}$ has the general form (2.10), constrained by the requirement that particle trajectories coincide with the string boundary. The resulting boundary conditions are interpreted as the particle equations of motion:

$$p_{\perp\lambda}{}^\mu p_{\perp\rho}{}^\nu \frac{DS^{\lambda\rho}}{ds} = 0 \,, \tag{5.10a}$$

$$\frac{D}{ds} \left(mv^\mu - 2v_\nu \frac{DS^{\mu\nu}}{ds} \right) - v^\nu S^{\lambda\rho} R^\mu{}_{\nu\lambda\rho} = n_a m^{ab} u_b^\mu \,. \tag{5.10b}$$

Here, $p_{\perp\nu}{}^\mu \equiv \delta_\nu^\mu + v^\mu v_\nu$ is the orthogonal projector to the string boundary, and should not be confused with $P_{\perp\nu}{}^\mu$. The boundary conditions (5.10) differ from the particle world line equations (5.1) by the presence of the string force on the right-hand side. As the boundary $\partial\mathcal{M}$ consists of two disjoint lines, the mass and spin of the two particles may differ.

In what follows, we shall assume that the string is made of the Nambu-Goto type of matter, moving in a 4-dimensional flat spacetime:

$$m^{ab} = T\gamma^{ab} \,, \qquad R^\mu{}_{\nu\lambda\rho} = 0 \,. $$

Then, the world sheet equations (5.9) reduce to the familiar Nambu-Goto equations, and the string force on the right-hand side of (5.10b) becomes Tn^μ. As for the particles, we shall impose the constraint

$$S^{\mu\nu} v_\nu = 0 \,, \tag{5.11}$$

which rules out the boost degrees of freedom. Physically, this condition constrains the particle centre of mass to coincide with the string end, with accuracy $\mathcal{O}_2$. After this, we are left with

$$\vec{S} \equiv \varepsilon^{\lambda\rho\mu 0} S_{\lambda\rho} \vec{e}_\mu$$

as the only independent components of $S^{\mu\nu}$.

Now, we look for a simple, *straight line* solution of the equations of motion (5.9). Without loss of generality, we put

$$\vec{z} = \vec{\alpha}(\tau)\sigma \,, \qquad z^0 = \tau \,,$$

where $\xi^0 \equiv \tau$ and $\xi^1 \equiv \sigma$ take values in the intervals $(-\infty, \infty)$ and $[-1, 1]$, respectively. Assuming that the string length $L = 2|\vec{\alpha}|$, and the velocity of the string ends $V = |d\vec{\alpha}/d\tau|$ are constant, the equation (5.9) reduces to

$$\frac{d^2}{d\tau^2}\vec{\alpha} + \omega^2\vec{\alpha} = 0 \,, \qquad \omega \equiv \frac{2V}{L} \,.$$

It describes uniform rotation in a plane. Choosing the rotation plane to be the $x - y$ plane, we get the solution

$$\vec{\alpha} = \frac{L}{2}\left(\cos\omega\tau\,\vec{e}_x + \sin\omega\tau\,\vec{e}_y\right). \tag{5.12}$$

Next we consider the boundary equations (5.10). Omitting the details of the calculation, we find that the particle intrinsic angular momentum satisfies

$$\frac{d\vec{S}}{d\tau} = 0 \,, \qquad \vec{S} = S\vec{e}_z \,, \tag{5.13}$$

while its velocity becomes

$$V = \frac{1}{\sqrt{1 + \frac{2\mu}{TL}}} \,, \qquad \mu \equiv m + \sqrt{\frac{2T}{mL}}S \,. \tag{5.14}$$

Each of the two particles has its own mass and intrinsic angular momentum, denoted by $m_\pm$ and $S_\pm$ for the particle at $\sigma = \pm 1$. As both particles have the same velocity, their masses are related by $\mu_+ = \mu_-$. We see that the particle masses $m_\pm$ may differ, in spite of the fact that the centre of mass of the string-particle system is at $\sigma = 0$. This is a consequence of the nontrivial spin-orbit interaction that contributes to the total energy.

By inspecting the expression (5.14), we see that $V < 1$, as it should be. In the limit $\mu \to 0$, the string ends move with the speed of light, representing the Nambu-Goto dynamics with Neumann boundary conditions. When $\mu \to \infty$, the string ends do not move. This is an example of Dirichlet boundary conditions.

The total angular momentum and energy of the considered system are calculated using (5.6) and (5.7). One finds

$$E = TL\frac{\arcsin V}{V} + \frac{2\mu}{\sqrt{1 - V^2}} - \frac{2V}{L}\left(S_+ + S_-\right),$$

$$J = \frac{TL^2}{4}\left(\frac{\arcsin V}{V^2} - \frac{\sqrt{1 - V^2}}{V}\right) + \frac{2\mu}{\sqrt{1 - V^2}}\frac{LV}{2} + S_+ + S_- \,.$$

These equations have obvious interpretation. The total energy of the system consists of the string energy, kinetic energy of the two particles, and the spin-orbit interaction energy. The particle intrinsic rotational energy, being quadratic in $\vec{S}$, is neglected in the pole-dipole approximation. The total angular momentum includes the orbital angular momentum of the string and the two particles, and the particle spins.

In the limit of small particle masses, the free parameter L can be eliminated in favour of E, which leads to

$$J = \frac{1}{2\pi T} E^2 + 2 \left(S_+ + S_- \right).$$

The first term on the right-hand side defines the known Regge trajectory, while the second represents a small correction due to the presence of spinning particles at the string ends. As we can see, the unique Regge trajectory of the ordinary string theory splits into a family of distinctive trajectories.

6. Concluding remarks

The study in the preceding sections concerns the dynamics of classical brane-like matter in curved backgrounds. In the simple case we have considered, the target space geometry is Riemannian. The type of matter fields is not specified. We only assume that matter fields are sharply localized around a brane.

The method we use is a generalization of the Mathisson-Papapetrou method for point-like matter [2, 3]. It has already been used in [4] for the study of infinitely thin string. In this work, higher branes are considered in the approximation where nonzero thickness of the brane is taken into account. As a consequence, additional degrees of freedom appear to characterize the intrinsic angular momentum of the brane.

The results of our analysis can be summarized as follows. In section 2 we have refined the Mathisson-Papapetrou method by developing a manifestly covariant decomposition of the stress-energy tensor into a series of δ-function derivatives. The truncation of the series at any level has been proven invariant with respect to both, spacetime and world sheet diffeomorphisms. We have also utilized two extra gauge symmetries. In particular, the extra symmetry 2 has been used to properly define the central surface of branelike mass distribution.

In section 3 the p-brane world sheet equations and boundary conditions have been derived in the pole-dipole approximation. Beside the effective stress-energy tensor of the brane, a new set of coefficients appear to characterize the nonzero brane thickness. They have been interpreted in section 4 as effective brane currents associated with the intrinsic angular momentum of the brane. By the proper definition of the central surface of mass, we have shown that charges associated with boosts can be gauged away.

Finally, we provided some examples. A particularly interesting one is a spinless string with spinning particles attached to its ends. When applied to the Nambu-Goto matter, it gives the correction to the behavior of the known Regge trajectories.

Let us say, in the end, that these results can be generalized to include the effects of the background torsion. The brane dynamics in the Riemann-Cartan spacetimes will be the objective of our next paper.

Acknowledgments

A. Differential geometry of surfaces

Throughout the paper we deal with the geometry of surfaces embedded in a general Riemannian spacetime. Here we introduce some basic notions needed for the exposition.

Consider a D-dimensional Riemannian spacetime parametrized by the coordinates x^μ. The metric tensor is denoted by $g_{\mu\nu}(x)$ and has Minkowski signature $\mathrm{diag}(-,+,\dots,+)$. Given the metric, one introduces the Levi-Civita connection

$$\Gamma^\mu{}_{\rho\sigma} \equiv \frac{1}{2} g^{\mu\lambda} \left(\partial_\rho g_{\lambda\sigma} + \partial_\sigma g_{\lambda\rho} - \partial_\lambda g_{\rho\sigma} \right) ,$$

and the covariant derivative ∇_λ:

$$\nabla_\lambda V^\mu \equiv \partial_\lambda V^\mu + \Gamma^\mu{}_{\rho\lambda} V^\rho .$$

The Riemann curvature tensor is defined as

$$R^\mu{}_{\lambda\nu\rho} \equiv \Gamma^\mu{}_{\lambda\rho,\nu} - \Gamma^\mu{}_{\lambda\nu,\rho} + \Gamma^\mu{}_{\sigma\nu}\Gamma^\sigma{}_{\lambda\rho} - \Gamma^\mu{}_{\sigma\rho}\Gamma^\sigma{}_{\lambda\nu} .$$

Now introduce a $(p+1)$-dimensional surface $\mathcal{M}$, parametrized by the coordinates ξ^a. If the surface equation is $x^\mu = z^\mu(\xi)$, one introduces the coordinate vectors

$$u_a^\mu \equiv \frac{\partial z^\mu}{\partial \xi^a} .$$

The induced metric tensor on the surface is defined by

$$\gamma_{ab} = g_{\mu\nu}(z) u_a^\mu u_b^\nu .$$

Assume that the surface is everywhere regular and the coordinates well defined. The induced metric is nondegenerate and of Minkowski signature $\mathrm{diag}(-,+,\dots,+)$.

Given an arbitrary spacetime vector V^μ, one can uniquely split it into a sum of vectors orthogonal and tangential to the surface $\mathcal{M}$, $V^\mu = V_\perp^\mu + V_\parallel^\mu$, using the projectors

$$P_\parallel{}^\mu{}_\nu \equiv u_a^\mu u_\nu^a , \qquad P_\perp{}^\mu{}_\nu \equiv \delta_\nu^\mu - u_a^\mu u_\nu^a ,$$

so that $V_\perp^\mu = P_\perp{}^\mu{}_\nu V^\nu$ and $V_\parallel^\mu = P_\parallel{}^\mu{}_\nu V^\nu$. Next, one can define the induced connection $\Gamma^a{}_{bc}$ by parallel transporting a vector using the spacetime connection, and then projecting the result onto the surface $\mathcal{M}$. If the connection is defined this way, one can show that it is precisely the Levi-Civita connection

$$\Gamma^a{}_{bc} = \frac{1}{2} \gamma^{ad} \left(\partial_b \gamma_{dc} + \partial_c \gamma_{db} - \partial_d \gamma_{bc} \right) .$$

Now, one can define the *total* covariant derivative ∇_a, that acts on both types of indices:

$$\nabla_a V^{\mu b} = \partial_a V^{\mu b} + \Gamma^\mu{}_{\lambda\rho} u_a^\rho V^{\lambda b} + \Gamma^b{}_{ca} V^{\mu c} .$$

The metricity conditions $\nabla_a g_{\mu\nu} = \nabla_a \gamma_{bc} = 0$ are identically satisfied. One can also introduce the second fundamental form,

$$K^\mu_{ab} \equiv \nabla_a u^\mu_b \,,$$

which satisfies the useful identities:

$$K^\mu_{ab} = K^\mu_{ba}, \qquad K^\mu_{ab} u^c_\mu = 0 \,.$$

The surface $\mathcal{M}$ may have a boundary $\partial\mathcal{M}$, and we denote its coordinates by λ^i. The boundary is supposed to satisfy the analogous geometric requirements as the surface itself. Given the boundary $\xi^a = \zeta^a(\lambda)$, one introduces its coordinate vectors

$$v^a_i \equiv \frac{\partial \zeta^a}{\partial \lambda^i} \,,$$

and the induced metric

$$h_{ij} = \gamma_{ab}(\zeta) v^a_i v^b_j \,.$$

The induced connection $\Gamma^i{}_{jk}$ is the Levi-Civita connection, so that the total covariant derivative ∇_i, which acts as

$$\nabla_i V^{\mu bj} = \partial_i V^{\mu bj} + \Gamma^\mu{}_{\lambda\rho} v^\rho_i V^{\lambda bj} + \Gamma^b{}_{ca} v^a_i V^{\mu cj} + \Gamma^j{}_{ki} V^{\mu bk} \,,$$

satisfies the metricity conditions $\nabla_i g_{\mu\nu} = \nabla_i \gamma_{ab} = \nabla_i h_{jk} = 0$. Here, $v^\mu_i \equiv u^\mu_a v^a_i$ are the spacetime components of the boundary coordinate vectors. The boundary projectors are defined as $p_\parallel{}^\mu_\nu \equiv v^\mu_i v^i_\nu$ and $p_\perp{}^\mu_\nu \equiv \delta^\mu_\nu - v^\mu_i v^i_\nu$.

Throughout the paper, the covariant form of the Stokes theorem is used:

$$\int_\mathcal{M} d^{p+1}\xi \sqrt{-\gamma} \nabla_a V^a = \int_{\partial\mathcal{M}} d^p\lambda \sqrt{-h} \, n_a V^a \,.$$

Here, n_a is the normal to the boundary. It is defined as

$$n_a = \frac{1}{p!} e_{ab_1...b_p} e^{i_1...i_p} v^{b_1}_{i_1} \ldots v^{b_p}_{i_p} \,,$$

where $e_{ab_1...b_p}$ and $e^{i_1...i_p}$ are totally antisymmetric world tensors on the surface and the boundary, respectively. They are defined using the Levi-Civita symbols $\varepsilon_{ab_1...b_p}$ and $\varepsilon^{i_1...i_p}$, and corresponding metric determinants:

$$e_{ab_1...b_p}(\xi) \equiv \sqrt{-\gamma} \, \varepsilon_{ab_1...b_p}, \qquad e^{i_1...i_p}(\lambda) \equiv \frac{1}{\sqrt{-h}} \varepsilon^{i_1...i_p} \,.$$

The normal n_a is always spacelike, and satisfies the following identities:

$$n_a n^a = 1 \,, \qquad n_a v^a_i = 0 \,, \qquad P_\perp{}^\mu_\nu = p_\perp{}^\mu_\nu - n^\mu n_\nu \,,$$

where $n^\mu \equiv u^\mu_a n^a$.

References

[1] H. Nielsen and P. Olesen, *Vortex line models for dual strings*, *Nucl. Phys.* **B 61** (1973) 45.

[2] M. Mathisson, *Neue Mechanik materieller Systeme*, *Acta Phys. Polon.* **6** (1937) 163.

[3] A. Papapetrou, *Spinning test particles in general relativity. 1*, *Proc. R. Soc.* **A209** (1951) 248.

[4] M. Vasilić and M. Vojinović, *Classical string in curved backgrounds*, *Phys. Rev.* **D 73** (2006) 124013, gr-qc/0610014.

[5] P. Yasskin and W. Stoeger, *Propagation equations for test bodies with spin and rotation in theories of gravity with torsion*, *Phys. Rev.* **D 21** (1980) 2081.

[6] K. Nomura, T. Shirafuji and K. Hayashi, *Spinning test particles in space-time with torsion*, *Prog. Theor. Phys.* **86** (1991) 1239.